Wave Superposition Solutions

1. The waves can add constructively to produce a total amplitude of 7, or destructively for a total amplitude of 1.

2. 
   a) We know that when waves of equal amplitude add, the resulting amplitude is \( 2A \cos(\phi/2) \), where \( \phi \) is the phase difference between the two. We also know that the phase difference due to waves of different frequency is \( \phi = -t \Delta \omega \). So we see that the total amplitude changes with time: 
   \[
   A_f = 2A \cos(-t \Delta \omega/2) = 2A \cos(t \Delta \omega/2).
   \]
   We are interested in constructive interference. This occurs when the resulting amplitude is either \(-2A\) or \(+2A\). So
   \[
   2A \cos(t \Delta \omega/2) = \pm 2A \Rightarrow \cos(t \Delta \omega/2) = \pm 1.
   \]
   \[
   \cos(x) = \pm 1 \text{ when } x = \frac{n \pi}{2} \text{ for } n \text{ even, i.e., } n = 0, \pm 2, \pm 4, \text{ etc.}
   \]
   So 
   \[
   t = \frac{n \pi}{2 \Delta \omega} \Rightarrow t = \frac{n \pi}{2} \text{ for } n = 0, \pm 2, \pm 4, \text{ etc.}
   \]
   
   \[
   t_{\text{beat}} = \frac{n \pi}{2 \Delta f} \text{ for } n = 0, \pm 2, \pm 4, \text{ etc.}
   \]
   b) If one beat occurs at \( t = \frac{n \pi}{2 \Delta f} \), the next will occur at \( t = \frac{n+2 \pi}{2 \Delta f} \). So the difference is 
   \[
   \Delta t = \frac{n+2 \pi}{2 \Delta f} - \frac{n \pi}{2 \Delta f} = \frac{2 \pi}{2 \Delta f} = \frac{\pi}{\Delta f}.
   \]
   So the beat period is 
   \[
   T_{\text{beat}} = \frac{1}{\Delta f}
   \]
   c) \( f = \frac{1}{T} \), so 
   \[
   f_{\text{beat}} = \frac{\Delta f}{\Delta f} = 888 \text{ Hz} - 879 \text{ Hz} = 4 \text{ Hz}
   \]

3. When \( f_{\text{piano}} = f_{\text{fiddle fork}} \), \( \Delta f = 0 \), so there will be no beats. If \( f_p \neq f_t \), then \( \Delta f \neq 0 \) and there will be beats. So listen for beats, and when they disappear, the two will be in tune.

4. 
   a) \( v = \lambda f \Rightarrow \lambda = \frac{v}{f} = \frac{340 \text{ m/s}}{100 \text{ Hz}} = 3.4 \text{ m} \)
   
   so 
   \[
   k = \frac{2\pi}{\lambda} = \frac{2\pi}{3.4} = \frac{10\pi}{17} \text{ m}^{-1} \approx 1.848 \text{ m}^{-1}
   \]
As before, \( A_{\text{tot}} = 2A \cos(\phi/2) \). In this case, the phase difference is due to position, so \( \phi = kAx = \Delta x/\lambda \).
So \( A_{\text{tot}} = 2A \cos(\Delta x/2\lambda) \).
\( \Delta x \) is the difference between the distances that separate you from each speaker. So distance to speaker 1 is \( x_1 = x \) and distance to speaker 2 is \( x_2 = l-x \).
So \( \Delta x = x_2-x_1 = l-x-x = l-2x \).
So \( A_{\text{tot}} = 2A \cos(l-2x/2\lambda) \).

For destructive interference, \( A_{\text{tot}} = 0 \Rightarrow \cos(l-2x/2\lambda) = 0 \).
So \( l-2x = n\pi \) for \( n = \pm 1, \pm 3, \ldots \), i.e., \( n \) odd.

Solving for \( x \): \( l-2x = n\pi \\lambda \Rightarrow 2x = l - n\pi \lambda \Rightarrow x = \frac{1}{2}(l-n\pi \lambda) \).
So we need to find the smallest allowed \( x \) such that \( 0 \leq x \leq l \) (otherwise, you wouldn't be between the speakers.)
Using \( l = 15 \text{ m} \) and \( \lambda = 3.4 \text{ m} \), we see that the first destructive interference occurs at \( x = 7.84 \text{ m} \) corresponding to \( n = -1 \). (Just plug in numbers)

For constructive interference, \( n \) must be even.

So by plugging in numbers, we see that there are 2 values of \( n \) that give \( 0 \leq x \leq l \): \( n = -2 \) and \( n = 0 \), corresponding to \( x = 13.181 \text{ m} \) and \( x = 2.5 \text{ m} \).

5. A guitar string is fixed at both ends, so it has displacement nodes at both ends. A tube with two open ends has pressure nodes at both ends, so a guitar string is like a tube with two open ends. (A tube with one closed end has one node and one antinode at its ends).
6. For a standing wave, the amplitude is given by
   \[ A_{\text{tot}} = 2A \sin(kx) \].
   In a cello string, both ends must be nodes. So
   \[ A_{\text{tot}}(x=0) = 0 \] and \[ A_{\text{tot}}(x=l) = 0 \] if the string has length \( l \).
   So \[ 2A \sin(k \cdot 0) = 0 \] and \[ 2A \sin(k \cdot l) = 0 \].
   \( 2A \sin(0) = 0 \) is not interesting because \( \sin(0) \) is always zero. But\( \sin(kl) = 0 \) only when \( kl = \frac{n\pi}{2} \) for \( n \) even.
   So \( k = \frac{n\pi}{2l} \) \( \Rightarrow \frac{2\pi}{\lambda} = \frac{n\pi}{2l} \) \( \Rightarrow \lambda = \frac{4l}{n} \) for \( n \) even.
   Alternatively, \( \lambda = \frac{2l}{j} \) for \( j = 1, 2, 3, \) etc (do you see why?)
   The third harmonic corresponds to \( n = 6 \) or \( j = 3 \), so
   \[ \lambda = \frac{2 \cdot 0.5 m}{3} = \frac{1}{3} m \approx 0.33 \text{ m} \]

7. Remember that in a string, \( v = \sqrt{\frac{T}{\mu}} \), and that \( v = 2f \).
   So \( 2f = \sqrt{\frac{T}{\mu}} \) \( \Rightarrow f = \frac{1}{2} \sqrt{\frac{T}{\mu}} \). \( \lambda \) is determined by the length of the string, so it is a constant, as is \( \mu \). So adjusting \( T \) changes \( f \).

8. For an open tube (just as in a guitar string), \( \lambda = \frac{2l}{j} \) for \( j = 1, 2, 3, \) etc. So \( l = \frac{1}{2} j \cdot \frac{1}{2}(4m)(1) = \frac{2m}{l} \).
   So its second harmonic has \( j = 2 \), so \( \lambda = \frac{2 \cdot 2m}{2} = 2m \).

9. Touching the string at the middle forces it to have a node there. So the string goes from
   \( \bigcirc \) to \( \bigcirc \), so its wavelength shrinks by a factor of two. Since \( f = \frac{1}{2} \), frequency will double.