Oscillatory Motion

1 Topics Covered

- If we have the ordinary differential equation (ODE) $\frac{d^2x}{dt^2} + \omega^2 x = 0$, the solution is given by $x = A \cos(\omega t + \phi)$.
- Any body whose motion can be described by this ODE is called a simple harmonic oscillator (SHO) and its motion is called simple harmonic motion.
- The angular frequency of the oscillations is given by $\omega$, the period of the oscillations is given by $T = 2\pi/\omega$, and the “regular” frequency is given by $f = 1/T = \omega/2\pi$.
- This ODE turns up whenever there is a linear restoring force ($F = -\alpha x$) or a quadratic potential ($U = \beta x^2$), and many real-life situations can be approximated by this situation.
- Spring: suppose an object of mass $m$ is on a massless spring of spring constant $k$ that obeys Hooke’s Law. Then the object is a SHO with $\omega^2 = k/m$.
- Physical pendulum: suppose an object of mass $m$ hangs on a rigid “rod” and the moment of inertia of the whole system is $I$. If the object swings back and forth with small displacements, it is a SHO with $\omega^2 = mgd/I$, where $d$ is the distance from the pivot to the center of mass of the whole system.
- Torsional pendulum: suppose a symmetric object (or group of objects) with moment of interia $I$ is connected to a wire of torsional constant $\kappa$ at its center of mass. Then if the object spins with small angular displacements, it is a SHO with $\omega^2 = \kappa/I$.
- Suppose an object of mass $m$ is on a massless spring of spring constant $k$ that obeys Hooke’s Law, but that there is also a frictional force proportional to velocity ($f_f = -bv$). Then we have the ODE $m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0$. Its solution is $x = Ae^{-bt/2m} \cos(\omega' t + \phi)$, with $\omega'^2 = k/m - b^2/4m^2$. (Note that the angular frequency decreases.) This is called damped harmonic motion.
- If $\omega'^2 > 0$, we say that the spring is underdamped; if $\omega'^2 = 0$, we say that the spring is critically damped; if $\omega'^2 < 0$, we say that the spring is overdamped.
- If there is a force $F$ pushing the spring, it is called driven harmonic motion and we have the ODE $m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = F$. The amplitude of the resulting function for $x$ has a maximum when $F$ has the same frequency as the natural frequency of the system (i.e., the frequency it would have without damping). This phenomenon is called resonance.

2 Problems

1. Suppose you have an object at the end of a spring with $k = 400$ N/m. If the spring undergoes 20 full oscillations in 30 s, what is the mass of the object?

2. Suppose you have an object of mass $m = 0.3$ kg at the end of a spring with $k = 20$ N/m. The object is in simple harmonic motion with $A = 0.1$ m.

   (a) What is the total energy of the system? How is it related to the energy stored in the spring? Does this make sense?
(b) What is the energy stored in the spring when the object’s velocity is 0.5 m/s?
(c) What is the object’s position when its velocity is 0.5 m/s?

3. Consider an object of mass $m$ on a spring with spring constant $k$. Suppose that $x = 0$ is the equilibrium position and that the object’s maximum displacement is $x = A$.

(a) At which value(s) of $x$ is the magnitude of the object’s velocity at a maximum?
(b) At which value(s) of $x$ is the magnitude of the object’s acceleration at a maximum?
(c) At which value(s) of $x$ is the magnitude of the object’s velocity at a minimum?
(d) At which value(s) of $x$ is the magnitude of the object’s acceleration at a minimum?

4. Derive the frequency of an ideal pendulum from the physical pendulum formula.

5. Suppose I have an object of mass $m = 0.4$ kg at the end of a spring with $k = 40$ N/m. Suppose also that there is a frictional force with constant $b = 0.1$ Ns/m.

(a) Is the system underdamped, overdamped, or critically damped?
(b) What value of $b$ corresponds to critical damping?
(c) If I add a driving force, what frequency should it have to maximize the amplitude of the oscillations?

6. Verify that $x(t) = e^{i\omega t}$ is a solution to the ODE $\frac{d^2x}{dt^2} + \omega^2 x = 0$, where $i = \sqrt{-1}$.

7. Suppose that $x(t) = A \cos(\omega t) + B \sin(\omega t)$, that $x(0) = x_0$, and that $x'(0) = v_0$. Find $A$ and $B$ in terms of $x_0$ and $v_0$. 

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