I. Sets in General:
   A. Subset: A is a subset of B iff every element of A is also an element of B
   B. Equality: A = B iff A is a subset of B and B is a subset of A
   C. Power Set: The power set of A is the set whose elements are subsets of A, i.e. 
      \{B: B is a subset of A\}
   D. Union: The union of A and B is \{x: x is an element of A or x is an element of B\}
   E. Intersection: The intersection of A and B is \{x: x is an element of A and x is an element of B\}
   F. Difference: The difference of A and B is the set A – B = \{x: x is an element of A and x is not an element of B\}
   G. Disjoint: A and B are disjoint iff their intersection is the empty set
   H. Complement: If U is the universe and B is a subset of U, then we define the 
      complement of B to be U – B
   I. Union over A: The union over A is \{x: x is an element of A for some set A 
      which is an element of A\}
   J. Inductive: A set S which is a subset of the set of all natural numbers is 
      inductive iff for every natural number n, n is element of S implies that n+1 is 
      an element of S

The PMI/WOP:
II. Definition of the Principle of Mathematical Induction:
   A. Definition One:
      1. Let P(n) be a statement about a natural number n
      2. If:
         i. P(1) is true
         ii. For every natural number n, P(n) implies P(n+1)
      3. Then: For every natural number n, P(n)
   B. Definition Two:
      1. Let S be a set
      2. If:
         i. S is a subset of the set of all natural numbers
         ii. S is inductive
         iii. 1 is an element of S
      3. Then: S is the set of all natural numbers

III. Using the PMI:
   A. The theorem describes a property that every natural number has
   B. So, say, “let P(n) be the statement (the property from the theorem states)”
   C. Show that P(1) is true
   D. Let nbar be arbitrary
   E. Assume P(nbar) and arrive at P(nbar+1)
   F. So P(nbar) implies P(nbar+1)
   G. So for every n, P(n) implies P(n+1)
   H. Since P(1) and for every n, P(n) implies P(n+1), the PMI says, for every n, 
      P(n)
IV. The Well-Ordering Principle (WOP):
A. If set S is a nonempty subset of the natural numbers, then S has a smallest element.
B. The WOP is a consequence of the PMI

V. Proving the WOP from the PMI:
A. Assume PMI is true
B. Suppose T is an arbitrary nonempty subset of the natural numbers
C. Let S = the set of natural numbers – T
D. Since T is not empty, S is not equal to the set of natural numbers
E. Suppose T has no smallest element
F. 1 is the smallest natural number, so 1 cannot be in T. So 1 is in S.
G. Suppose n is an element of S
H. No number less than n is in T because one of them would have to be the smallest element of T
I. n is not an element of T because n is an element of S
J. So n+1 is not an element of T because it would be the smallest member of T. So n+1 is in S
K. Since 1 is in S and n in S implies n+1 in S, every natural number is in S (by the PMI)
L. This contradicts the fact that S is not the set of natural numbers
M. So T has a smallest element. So every nonempty subset of the natural numbers has a smallest element

VI. Using the WOP for Fibonacci number proofs:
A. The proof will be some statement F(n) about Fibonacci numbers
B. Make a set of “Bad” numbers, i.e., natural numbers that do not meet this property
C. Assume that it is not empty
D. Apply the WOP to get a smallest element b
E. Test by case b = 1 and b = 2. So you know that b is greater than 2.
F. Then you know that anything smaller than b is not is B, so the property is true for b - 1 and b – 2, which are both natural numbers. Use this to arrive at a contradiction.
G. So B is empty
H. So every natural number is not in B
I. So every natural number has property F(n)

Relations:
VII. Relations in General:
A. Ordered Pair: a pair of coordinates, (a,b)
   1. Alternate definition: (a,b) = {{a},{a,b}}
B. Equality of Ordered Pairs: (a,b) = (c,d) iff a = c and b = d
C. Cartesian Product: The Cartesian Product of A and B is written as AxB and is given by AxB = {(a,b): a is an element of A and b is an element of B}
D. Relation: R is a relation from A to B iff R is a subset of AxB
E. R-Related: If \((a,b)\) is an element of \(R\), we write \(aRb\) and say that \(a\) and \(b\) are \(R\)-related.

F. Relation on \(A\): A relation \(R\) from \(A\) to \(A\) is called a relation on \(A\).

G. Inverse: If \(R\) is a relation from \(A\) to \(B\), then the inverse of \(R\) is \(R^{-1} = \{(y,x) : (x,y)\ is\ an\ element\ of\ R\}\).

H. Composite: If \(R\) is a relation from \(A\) to \(B\) and \(S\) is a relation from \(B\) to \(C\), then the composite of \(R\) and \(S\) is \(S'R = \{(a,c) : \text{there exists an element } b \text{ of } B \text{ such that } aRb \text{ and } bSc\}\).

I. Domain: If \(R\) is any relation, then \(\{x : \text{there exists a } y \text{ such that } xRy\}\) is the domain of \(R\).

VIII. Equivalence Relations:

A. Equivalence Relation on \(A\): A relation \(R\) is an equivalence relation on \(A\) iff \(R\) is reflexive on \(A\), symmetric, and transitive.

B. Reflexive: \(R\) is reflexive on \(A\) iff for every \(x\) in \(A\), \(xRx\).

C. Symmetric: \(R\) is symmetric iff for every \(x\) and \(y\), \(xRy\) implies \(yRx\).

D. Transitive: \(R\) is transitive iff for every \(x\), \(y\), and \(z\), \(xRy\) and \(yRz\) imply \(xRz\).

E. Equivalence Class: Let \(R\) be an equivalence relation on \(A\). For \(x\) which are in \(A\), the equivalence class of \(x\) determined by \(R\) is the set \(x/R = \{y : y\ is\ an\ element\ of\ A : xRy\}\).

F. \(x\) modulo \(R\): \(x/R\) is read as “\(x\) modulo \(R\)”

G. \(A\) modulo \(R\): The set of all equivalence classes is called \(A\) modulo \(R\) and is denoted by \(A/R = \{x/R : x\ is\ an\ element\ of\ A\}\).

IX. Congruence modulo \(m\):

A. Definition 1: for any natural number \(m\), and for any integers \((x,y)\), \(x\) is congruent modulo \(m\) to \(y\) iff they have the same remainder when divided by \(m\).

B. Definition 2: For any natural number \(m\), and for any integers \((x,y)\), \(x\) is congruent modulo \(m\) to \(y\) iff \(m\) divides \(x – y\).

X. Outline for proving that Definition 1 is equivalent to Definition 2:

A. Use the division algorithm to divide \(x\) and \(y\) by \(m\).

B. Part 1: Assume Definition 1. So set the remainders equal and subtract \(y\) from \(x\). Show that \(m\) divides \(x – y\).

C. Part 2: Assume Definition 2. So \(m\) divides \(x – y\). So \(x = y + km\). Replace \(y\) with its division algorithm form, and then get a second equation for \(x\) by setting it equal to its division algorithm form. Show that the remainders are equal. (Because DA remainders are unique for each divisor.)

XI. Trick for proving that “\((m,n)R(m`,n`)\) iff \(mn` = m`n\)” is transitive: Multiply the equation from the first assumed relation by the second coordinate in the third ordered pair.

Partitions:

XII. Partition: A partition of a nonempty set \(A\) is a set \(A\) such that:

1. \(A\) is a subset of the power set of \(A\).
2. Every element of \(A\) is not the empty set.
3. If \(X\) and \(Y\) are elements of \(A\), then either \(X = Y\) or \(X\) and \(Y\) are disjoint.
4. Every element of \(A\) is a member of some element of \(A\).
XIII. Outline of proving that for an equivalence relation \( R \) on \( A \), \( A \) modulo \( R \) is a partition on \( A \):

A. Let:
   1. \( A \) be an arbitrary nonempty set
   2. \( R \) be an arbitrary equivalence relation on \( A \)
   3. \( x/R = \{ y \in A : yRx \} \), for \( x \in A \)
   4. \( A = A/R = \{ x/R : x \text{ is an element of } A \} \)

B. Step One: Prove that \( A \) is a subset of the power set of \( A \)
   1. Make an arbitrary set \( B \), and assume that it is an element of \( A \)
   2. Pick a \( b \) such that \( B = b/R \). So \( B \) is not empty because \( b \) is in \( B \).
   3. Make an arbitrary \( z \) and assume it is an element of \( B \)
   4. So \( bRz \), so \( (b,z) \) is an element of \( R \). \( R \) is a subset of \( AxA \), so \( (b,z) \) is an element of \( AxA \), so \( z \) is an element of \( A \).
   5. Since \( z \) in \( B \) implies \( z \) in \( A \), \( B \) is a subset of \( A \), i.e., \( B \) is an element of the power set of \( A \)
   6. Since \( B \) in \( A \) implies \( B \) in the power set of \( A \), \( A \) is a subset of the power set of \( A \)

C. Step Two: Proving that no element of \( A \) is empty
   1. Make an arbitrary set \( B \), and assume that it is an element of \( A \)
   2. Pick a \( b \) such that \( B = b/R \). So \( B \) is not empty because \( b \) is in \( B \).
   3. Since \( B \) was arbitrary, ever element of \( A \) is not empty

D. Step Three: Proving that any two elements of \( A \) are either equal or disjoint
   1. Make arbitrary sets \( F \) and \( G \) and assume that they are elements of \( A \).
      Assume their intersection is not empty.
   2. So then pick a \( z \) that is an element of the intersection of \( F \) and \( G \)
   3. Pick \( f \) and \( g \) such that \( F = f/R \) and \( G = g/R \)
   4. Since \( z \) is in both sets, \( fRz \) and \( gRz \). Using symmetry and transitivity, show that \( fRg \) and \( gRf \)
   5. Make an arbitrary \( s \) and assume it is an element of \( F \). Show that it is an element of \( G \).
   6. Since \( s \) in \( F \) implies \( s \) in \( G \), \( F \) is a subset of \( G \)
   7. Make an arbitrary \( s \) and assume it is an element of \( G \). Show that it is an element of \( F \).
   8. Since \( s \) in \( G \) implies \( s \) in \( F \), \( G \) is a subset of \( F \)
   9. Since \( F \) and \( G \) are subsets of each other, they are equal.
   10. So the intersection of \( F \) and \( G \) being not empty implies \( F = G \), i.e., the intersection is empty or they are equal
   11. So \( F \) and \( G \) being in \( A \) implies either \( F \) and \( G \) are equal or disjoint. So then generalize to all sets \( F \) and \( G \)

E. Step Four: Proving that every element of \( A \) is in some element of \( A \)
   1. Make an arbitrary \( x \) in \( A \)
   2. So \( xRx \), by reflexivity
   3. So \( x \) is an element of \( X = x/R \)
   4. \( X \) is an element of \( A \), by definition
   5. So \( x \) is in an element of \( A \)
   6. Since \( x \) was arbitrary, every \( x \) in \( A \) is in a member of an element of \( A \)
F. Since all four conditions for a partition have been met, A is a partition on A

XIV. Old theorems:
A. Division Algorithm: If a and b are positive integers with b less than or equal to a, then there exists a natural number q and a non-negative integer r such that a = bq + r, where 0 is less than or equal to r, and r is less than b.
B. Euclid’s Algorithm: Let a and b be two positive integers with b less than or equal to a. Let d be GCD(b,a). Then:
   1. Part 1: You can successively apply the division algorithm to a and b to find a final remainder r such that r = d
   2. Part 2: The GCD of a and b may be written as a combination of a and b, i.e., d = ax + by for some integers x and y
C. The Fundamental Theorem of Arithmetic: Every natural number can be expressed uniquely as a product of primes

Assorted Proof Outlines:
XV. Proof that the square root of a prime number p is irrational:
A. Assume that the square root of the p is rational, and rewrite as a fraction of integers
B. Square both sides, and remove denominator
C. Make a set consisting of all natural numbers which, when multiplied by p, equal an integer squared.
D. Show that it is not empty
E. Apply WOP to get a natural number, which, when multiplied by p, yields the square of an integer
F. Show that p divides that integer squared
G. Subproof: Prove that p divides that integer (not squared)
   1. Assume p does not divide it
   2. So the GCD of p and that integer is 1
   3. Use Euclid’s Algorithm
   4. Arrive at the fact that p does divide that integer
   5. So p does not divide it implies that p does divide it, i.e., p divides it or p divides it, i.e., p divides it
H. Apply definition of “divides”
I. Square this, and in doing so, find a new element of the set
J. Show that this element is smaller than the previous smallest element.
K. This is a contradiction, so the square root of p is not rational.

XVI. Proof that every natural number has a prime factor:
A. Let n be an arbitrary natural number
B. Make a set S of all numbers that divide it
C. n is in this set, so it is not empty, so you can apply the WOP to get S’s smallest member p
D. suppose p is not prime
E. then p has a factor
F. show that there is another natural number, p’, which is in S but smaller than p
G. this is a contradiction, so p is prime
H. so every natural number has a prime factor

XVII. Proof that there are infinite primes:
A. Let \( p_n \), denote a prime number
B. Assume that there are a finite number of primes
C. So, \( S = \{p_1, p_2, p_3, \ldots p_n\} \), where \( n \) is some natural number, is the set of all prime numbers
D. Let \( M = (p_1)(p_2)(p_3)\ldots(p_n) \) and let \( m = M+1 \). So \( m - M = 1 \)
E. Every natural number greater than 1 has a prime factor
F. So \( m \) is divisible by some prime number \( q \)
G. Suppose \( q \) is an element of \( S \). So \( q/M \). But \( q/m \). \( m - M = 1 \). Show that \( q \) divides 1.
H. So \( t = 1/q \) for some integer \( t \)
I. Arrive at a contradiction \( (0 = 1) \)
J. So \( q \) is not an element of \( S \)
K. So \( q \) is a prime number but \( q \) is not a member of the set of all prime numbers
L. This is a contradiction, so the set of primes is infinite

Functions:

XVIII. Functions in general:
A. Function: A function \( f \) from \( A \) to \( B \) is a relation \( f \) from \( A \) to \( B \) such that:
   1. For every \( x \in A \) there is a \( y \in B \) such that \( x f y \)
   2. For every \( x \in A \), if \( x f y \) and \( x f z \), then \( y = z \)
B. If \( f \) is a function from \( A \) to \( B \), we write \( f: A \rightarrow B \)
C. If \( f: A \rightarrow B \) and \( a \in A \), then we write \( f(a) \) for the unique \( b \in B \) such that \( (a, b) \in f \);
   i.e., instead of \( (a, b) \in f \), we write \( f(a) = b \)
D. Restriction: if \( f \) is function from \( A \) to \( B \) and \( S \) is a subset of \( A \), we can define a function \( f/S \) by letting \( f/S = \) the intersection of \( f \) and the Cartesian Product of \( S \) and \( B \); i.e., \( f/S = \{(x, y) \in f: x \text{ is an element of } S\} \)
E. One-to-one: let \( f \) be a function from \( A \) to \( B \). we say that \( f \) is one-to-one iff \( f(x) = f(y) \) implies \( x = y \)
F. On to: let \( f \) be a function from \( A \) to \( B \). \( f \) is onto \( B \) iff for every \( b \in B \), there is an \( a \in A \) such that \( f(a) = b \)
G. Bijection: Let \( A \) and \( B \) be sets. A bijection from \( A \) to \( B \) is a function \( f: A \rightarrow B \)
   which is one-to-one and onto
H. Equivalent: Two sets \( A \), \( B \) are equivalent iff there exists a bijection from \( A \) to \( B \), and we write \( A \approx B \).

XIX. Special Properties of Functions (easy to prove):
A. The composite of two functions is a function
B. The composite of two onto functions is onto
C. The composite of two one-to-one functions is one-to-one
D. The composite of two bijections is a bijection
E. The inverse of a function \( f \) is a function iff \( f \) is a bijection
F. \( \approx \) is an equivalence relation
Cardinality

XX. Cardinality in General:
   A. $N_k$: If $k \in \mathbb{N}$, then $N_k = \{n \in \mathbb{N} : n \leq k\}$
   B. Finite: A set $S$ is finite iff $S = \emptyset$ or $S \approx N_k$ for some $k \in \mathbb{N}$.
   C. Infinite: A set $S$ is infinite iff it is not finite
   D. Cardinal number: Let $S$ be a finite set. If $S \approx N_k$ for some $k \in \mathbb{N}$, then $S$ has cardinal number $k$, or cardinality $k$. If $S = \emptyset$ then we say that $S$ has cardinal number 0, or cardinality 0.
   E. Denumerable: Let $S$ be a set. $S$ is denumerable iff $S \approx \mathbb{N}$. For a denumerable set $S$, we say $S$ has cardinal number $\aleph_0$ or cardinality $\aleph_0$
   F. Countable: A set $S$ is countable iff it is finite or denumerable
   G. Uncountable: A set $S$ is uncountable iff it is not countable

XXI. Special Properties of Cardinality (proved later):
   A. The cardinality of a finite set is unique
   B. A finite set is not equivalent to any of its proper subsets
   C. Every subset of a finite set is finite
   D. Every proper subset of a finite set has smaller cardinality
   E. The union of two finite sets is finite

Consequences of Functions and Cardinality:

XXII. Pigeon Hole Principle: for every $n \in \mathbb{N}$, if $r \in \mathbb{N}$ and $n > r$ and $f: \mathbb{N}_n \rightarrow \mathbb{N}_r$, then $f$ is not one-to-one
   A. Lemma for proving the Pigeon Hole Principle: If $r \in \mathbb{N}$ and $r > 1$ and $x \in \mathbb{N}_r$, then $\mathbb{N}_r - \{x\} \approx \mathbb{N}_{r-1}$.
   B. Proving Pigeon Hole Principle:
      1. Use induction on “if $r \in \mathbb{N}$ and $n > r$ and $f: \mathbb{N}_n \rightarrow \mathbb{N}_r$, then $f$ is not one-to-one” starting with $n = 2$
      2. P(2) is true because if $f: \{1,2\} \rightarrow \{1\}$, then $f(1) = 1$ and $f(2) = 1$, so $f(1) = f(2)$ but $1 \neq 2$, so $f$ is not one-to-one
      3. Assume P(n) for arbitrary $n \in \mathbb{N}$
      4. Suppose $r \in \mathbb{N}$ and $n+1 > r$ and $F: \mathbb{N}_{n+1} \rightarrow \mathbb{N}_r$
      5. Suppose $F$ is one-to-one
      6. Let $g$ be the restriction of $F$ to $\mathbb{N}_n$.
      7. So $g: \mathbb{N}_n \rightarrow \mathbb{N}_r$
      8. Let $x \in \mathbb{N}_n$ be arbitrary
      9. Suppose $g(x) = F(n+1)$. Get a contradiction because then $F(x) = F(n+1)$, but $x \neq n+1$, so $F$ is not one-to-one
     10. So $g(x) \neq F(n+1)$ for any $x \in \mathbb{N}_n$
     11. So $g: \mathbb{N}_n \rightarrow \mathbb{N}_r - \{F(n+1)\}$
     12. Suppose $g$ is not one-to-one. Then there exist $a,b \in \mathbb{N}_n$ such that $g(a) = g(b)$ and $a \neq b$. Get a contradiction because $F$ is not one-to-one.
     13. So $g$ is one-to-one
14. By the Lemma, $N_r - \{F(n+1)\} \approx N_{r-1}$
15. So there exists a bijection $h: N_r - \{F(n+1)\} \rightarrow N_{r-1}$
16. So $h^g: N_n \rightarrow N_{r-1}$
17. Since $g$ is one-to-one and $h$ is one-to-one, $h^g$ is one-to-one
18. Since $r < n+1$, $r-1 < n$
19. So $h^g: N_n \rightarrow N_{r-1}$ and $r-1 < n$, which contradicts the hypothesis of induction
20. So $F$ is not one-to-one. So $r \in N$ and $n+1 > r$ and $F: N_{n+1} \rightarrow N_r$ imply $F$ is not one-to-one. So for every $r$, $r \in N$ and $n+1 > r$ and $F: N_{n+1} \rightarrow N_r$ imply $F$ is not one-to-one, i.e., $P(n+1)$.
21. Finish out PMI proof

XXIII. Proof Outlines for Special Properties of Cardinality:

A. Proving that the cardinality of a finite set is unique:
   1. Make an arbitrary set with cardinality $k$ and cardinality $m$
   2. Suppose $k \neq m$. So one is larger. Pick $k$ to be larger.
   3. Show that there is a bijection from $N_k$ to $N_m$
   4. The Pigeon Hole Principle says this is impossible
   5. So $k = m$

B. Proving that a finite set is not equivalent to any of its proper subsets:
   1. Let $A$ be an arbitrary finite set and let $B$ be a proper subset of $A$
   2. So $A \approx N_k$ for some $k \in N$ or $A = \emptyset$, and $B$ is a subset of $A$, and $B \neq A$
   3. Suppose $A = \emptyset$. So $A$ has no elements, so $B$ does not exist. So $\sim A \approx B$
   4. Suppose $A \approx N_k$ for arbitrary $k \in N$. From $B \neq A$ and $B$ is a subset of $A$, show that $A$ is not a subset of $B$
   5. So there is an $x$ that is in $A$ but not in $B$
   6. Make a bijection $f: A \rightarrow N_k$ and make $g$ its restriction to $B$
   7. Show by contradiction that $g(b) \neq f(x)$ for any $x$
   8. So $g: B \rightarrow N_k - \{f(x)\}$
   9. Apply Lemma to show that $N_k - \{f(x)\} \approx N_{k-1}$
   10. Use it to make a bijection $h$
   11. So there is now (by composite) a bijection from $B$ to $N_{k-1}$
   12. So $B \approx N_{k-1}$, but $\sim N_k \approx N_{k-1}$ because the Pigeon Hole Principle says that this is impossible. So $\sim B \approx A$.

C. Proving that any subset of a finite set is finite:
   1. Let $A$ be an arbitrary finite set. So $A \approx N_k$ for some $k \in N$ or $A = \emptyset$.
   2. Let $B$ be a subset of $A$
   3. Suppose $A = \emptyset$. Then $B = \emptyset$, so $B$ is finite
   4. Suppose $A \approx N_k$ for some $k \in N$
   5. Then there is a bijection $f: A \rightarrow N_k$
   6. Suppose $A = B$. Then $g: B \rightarrow N_k$ is a bijection. So $B \approx N_k$. So $B$ is finite.
   7. Suppose $A \neq B$. Then there is an $x$ in $A$ but not in $B$.
   8. Let $h$ be the restriction of $f$ to $B$
9. Show that \( h(b) \neq f(x) \) for any \( b \in B \)
10. So \( h: B \to \mathbb{N}_k - \{f(x)\} \)
11. Show that \( h \) is one-to-one and onto by contradiction
12. So \( h \) is a bijection and \( B \approx \mathbb{N}_k - \{f(x)\} \)
13. By the Lemma, \( B \approx \mathbb{N}_{k-1} \)
14. So \( B \) is finite
15. So in all cases, \( B \) is finite, and so any subset of a finite set is finite

XXIV. Other Assorted Proofs:

A. Proving that \( \mathbb{N} \) is infinite:
   1. Suppose \( \mathbb{N} \) is finite
   2. Then \( \mathbb{N} \approx \mathbb{N}_k \) for some \( k \in \mathbb{N}_k \) or \( \mathbb{N} = \emptyset \)
   3. \( \mathbb{N} \neq \emptyset \) because \( 1 \in \mathbb{N} \)
   4. So \( \mathbb{N} \approx \mathbb{N}_k \) for some \( k \in \mathbb{N}_k \)
   5. So there exists a bijection \( f: \mathbb{N}_k \to \mathbb{N} \)
   6. Let \( n = f(1) + f(2) + \ldots + f(k) + 1 \)
   7. Then \( n \in \mathbb{N} \) but \( \neg n \in \mathbb{N}_k \) because \( n \) is larger than any element of \( \mathbb{N}_k \)
   8. So \( f \) is not onto. But \( f \) is onto because it is a bijection. This is a contradiction. So \( \mathbb{N} \) is not finite. So \( \mathbb{N} \) is infinite.

B. Proving that \( (0,1) \) is uncountable:
   1. Suppose \( (0,1) \) is finite
   2. Then \( (0,1) \approx \mathbb{N}_k \) for some \( k \in \mathbb{N} \) or \( (0,1) \approx \emptyset \)
   3. \( (0,1) \approx \emptyset \) because \( 0.5 \in (0,1) \)
   4. So \( (0,1) \approx \mathbb{N}_k \) for some \( k \in \mathbb{N} \)
   5. So there exists a bijection \( f: \mathbb{N}_k \to (0,1) \)
   6. Let \( a \) be the largest of \( f(1), f(2), f(3), \ldots, f(k) \)
   7. Let \( b = (a+1)/2 \). Then \( b \in (0,1) \) but \( f(j) \neq b \) for any \( j \in \mathbb{N}_k \) because \( b > \) the greatest \( f(k) \)
   8. So \( f \) is not onto, contradicting the fact that \( f \) is a bijection.
   9. So \( (0,1) \) is infinite
10. Suppose \( (0,1) \) is denumerable
11. So \( \mathbb{N} \approx (0,1) \)
12. So there exists a bijection \( g: \mathbb{N} \to (0,1) \)
13. So \( f(1) = 0.a_{11}a_{12}a_{13} \ldots \)
   (i) \( f(2) = 0.a_{21}a_{22}a_{23} \ldots \)
   (ii) \( f(3) = 0.a_{31}a_{32}a_{33} \ldots \)
14. etc, where \( a_{mn} \in \{0,1,2,3,4,5,6,7,8,9\} \) and all decimals are in normalized form
15. let \( b = 0.b_1b_2b_3b_4b_5b_6\ldots \) where \( b_m = 5 \), if \( a_{mn} \neq 5 \) and \( b_m = 3 \), if \( a_{mn} = 3 \)
16. then \( b \in (0,1) \) but \( b \neq f(j) \) for any \( j \in \mathbb{N} \), so \( f \) is not onto, contradicting the fact that \( f \) is a bijection. So \( (0,1) \) is not denumerable
17. Since \( (0,1) \) is not finite or denumerable, it is uncountable